A LAPLACE TRANSFORM TECHNIQUE FOR WEDGE SHAPED ISOREFRACTIVE REGIONS

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Many techniques have been proposed for studying wedge shaped regions: among them it is important to mention the Malyuzhinets approach [1], which is based on the Sommerfeld representation. This technique yields an elegant formal procedure for solving difficult problems, like the diffraction by wedges with given surface impedances. However, even if the Sommerfeld integral is a valid ansatz for representing the solutions of the wave equation in angular regions, the Laplace transform appears to be a more valid representation, because of its solid mathematical foundation. Some authors [2, 3] have shown that the Laplace transform technique may be alternative with respect to the Malyuzhinets approach, even if in some cases it is not so simple and elegant.

In this paper we propose a new technique, based on the Laplace representations of the electromagnetic field, for solving isorefractive angular regions (Fig. 1) excited by an incident E-polarized plane wave in the z-direction. The technique can be briefly summarized as follows.

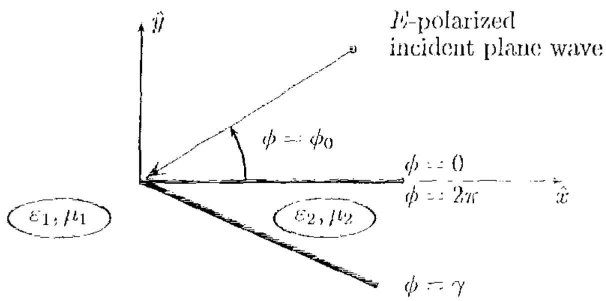

Figure 1: Geometry of the problem under investigation

By introducing the Laplace transform of the E_z and H_ρ components of the electromagnetic field

$$V(s,\phi) = \int_0^\infty E_z(\rho,\phi)\exp(-s\rho)d\rho \tag{1}$$

$$I(s,\phi) = \int_0^\infty H_\rho(\rho,\phi)\exp(-s\rho)d\rho \tag{2}$$

it is shown that in every angular homogeneous region the following representations hold

$$k\sin(w)V(s,\phi)|_{s=-jk\cos(w)} = A(w+\phi)+B(w-\phi) \tag{3}$$

$$ZI(s,\phi)|_{s=-jk\cos(w)} = A(w+\phi)-B(w-\phi) \tag{4}$$

where $k = \sqrt{\varepsilon_1\mu_1} = \sqrt{\varepsilon_2\mu_2}$ is the propagation constant of the isorefractive regions, $Z = 1/Y = \sqrt{\mu/\varepsilon}$ is the relevant impedance, and $A(w)$, $B(w)$ are suitable unknown functions.

By imposing the boundary conditions at the two interfaces $\phi = 0 - 2\pi$ and $\phi = \gamma$, the following system of linear difference equations is obtained

$$A_1(w) + B_1(w) = A_2(w + 2\pi) + B_2(w - 2\pi) \tag{5}$$

$$Y_1[A_1(w) - B_1(w)] = Y_2[A_2(w + 2\pi) - B_2(w - 2\pi)] \tag{6}$$

$$A_1(w + \gamma) + B_1(w - \gamma) = A_2(w + \gamma) + B_2(w - \gamma) \tag{7}$$

$$Y_1[A_1(w + \gamma) - B_1(w - \gamma)] = Y_2[A_2(w + \gamma) - B_2(w - \gamma)] \tag{8}$$

By multiplying both the sides of (5) by Y_2 and by summing and subtracting (6) from the resulting equation, we have

$$2Y_2A_2(w + 2\pi) = (Y_2 + Y_1)A_1(w) + (Y_2 - Y_1)B_1(w) \tag{9}$$

$$2Y_2B_2(w - 2\pi) = (Y_2 - Y_1)A_1(w) + (Y_2 + Y_1)B_1(w) \tag{10}$$

Then, the explicit solution with respect to A_2 and B_2 yields

$$A_2(w) = \frac{1 + Z_2Y_1}{2}A_1(w - 2\pi) + \frac{1 - Z_2Y_1}{2}B_1(w - 2\pi) \tag{11}$$

$$B_2(w) = \frac{1 - Z_2Y_1}{2}A_1(w + 2\pi) + \frac{1 + Z_2Y_1}{2}B_1(w + 2\pi) \tag{12}$$

Finally the substitution of expressions (11) and (12) into (7) and (8) allows to write the following homogeneous system, involving only the unknown functions $A_1(w)$ and $B_1(w)$

$$\begin{aligned} A_1(w + \gamma) + B_1(w - \gamma) &= \frac{1 + Z_2Y_1}{2}A_1(w + \gamma - 2\pi) + \frac{1 - Z_2Y_1}{2}B_1(w + \gamma - 2\pi) \\ &+ \frac{1 + Z_2Y_1}{2}A_1(w - \gamma + 2\pi) + \frac{1 - Z_2Y_1}{2}B_1(w - \gamma + 2\pi) \end{aligned} \tag{13}$$

$$\begin{aligned} Y_1[A_1(w + \gamma) - B_1(w - \gamma)] &= \frac{Y_1 + Y_2}{2}A_1(w + \gamma - 2\pi) + \frac{Y_2 - Y_1}{2}B_1(w + \gamma - 2\pi) \\ &- \frac{Y_2 + Y_1}{2}A_1(w - \gamma + 2\pi) - \frac{Y_2 - Y_1}{2}B_1(w - \gamma + 2\pi) \end{aligned} \tag{14}$$

At the interface $\phi = 0$, the longitudinal component of the electric and the magnetic fields can be written as the sum of the geometrical and the diffracted fields (E^d, H^d) as follows

$$E_z(\rho, 0) = A_0 \exp[jk\rho\cos(\phi_0)] + E_z^d(\rho) \tag{15}$$

$$H_\rho(\rho, 0) = Y_1A_0\sin(\phi_0)\exp[jk\rho\cos(\phi_0)] + H_\rho^d(\rho) \tag{16}$$

The corresponding Laplace Transforms $V(s, 0)$ and $I(s, 0)$ evaluated for $s = -jk\cos(w)$ take the form

$$
\begin{aligned}
k\sin(w)V(s,0)|_{s=-jk\cos(w)} &= k\sin(w)\int_0^\infty E_z(\rho,0)\exp(-s\rho)d\rho\Big|_{s=-jk\cos(w)} \\
&= \frac{j\sin(w)A_0}{\cos(w)+\cos(\phi_0)} + X(w) \\
I(s,0)|_{s=-jk\cos(w)} &= \int_0^\infty H_\rho(\rho,0)\exp(-s\rho)d\rho\Big|_{s=-jk\cos(w)} \\
&= Y_1[\frac{j\sin(\phi_0)A_0}{\cos(w)+\cos(\phi_0)} + Y(w)]
\end{aligned} \tag{17}
$$

where $X(w)$ and $Y(w)$ represent the Laplace Transforms (evaluated for $s = -jk\cos(\phi)$) of the diffracted electric and magnetic fields respectively, multiplied by Z_1.

Due to the boundary conditions at $\phi = 0$, equations (3) and (4) yield

$$
\begin{aligned}
\frac{j\sin(w)A_0}{\cos(w)+\cos(\phi_0)} + X(w) &= A_1(w) + B_1(w) \\
\frac{j\sin(\phi_0)A_0}{\cos(w)+\cos(\phi_0)} + Y(w) &= A_1(w) - B_1(w)
\end{aligned} \tag{18}
$$

By deriving from (18) the explicit expressions of $A_1(w)$ and $B_1(w)$ in terms of $X(w)$ and $Y(w)$ and by substituting such expressions in (13) and (14), we obtain a difference equation system which involves only the two unknowns $X(w)$ and $Y(w)$.

The advantage of addressing this system instead of (13)-(14) is that, owing to physical considerations, it is readily derived that the unknowns $X(w)$ and $Y(w)$ do not exhibit poles in the strip of complex w plane, within the interval $0 \le Re(w) < 2\pi$. This allows to solve the system of difference equations containing $X(w)$ and $Y(w)$ by using the Fourier Transform approach introduced in [1]

$$
\tilde{X}(\nu) = \mathcal{F}[X(w)] = \int_{-j\infty}^{j\infty} X(w)\exp(j\nu w)dw; \qquad \tilde{Y}(\nu) = \mathcal{F}[Y(w)] = \int_{-j\infty}^{j\infty} Y(w)\exp(j\nu w)dw \tag{19}
$$

that, owing to the pole location, satisfies the property

$$
\mathcal{F}[X(w+w_0)] = \tilde{X}(\nu)\exp(-j\nu w_0); \qquad \mathcal{F}[Y(w+w_0)] = \tilde{Y}(\nu)\exp(-j\nu w_0) \tag{20}
$$

The mathematical derivation of the unknowns, first in the spectral and then in the natural domain, presents several details and some delicate aspects, that, for lack of space, cannot be discussed here. They will be outlined and dealt with during the conference presentation.

References

[1] G. D. Maliuzhinets, "Excitation, Reflection, and emission of surface waves from a wedge with given face impedances," *Sov. Phys. Dokl, Eng. Trans.*, vol. 3, pp. 752-755, 1958.

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